Detailed Solutions – Calculus (Graduation Level)

Problem 1

$f(x) = \int_0^{x^2} \ln(1 + t^3)\, dt$

Find f'(x) and discuss whether f(x) is increasing or decreasing on [0, ∞).

Solution:

Let $F(u) = \int_0^u \ln(1 + t^3)\, dt$.
Then, by the Fundamental Theorem of Calculus (FTC), $F'(u) = \ln(1 + u^3)$.
Now, $f(x) = F(x^2) \Rightarrow f'(x) = F'(x^2) \times d(x^2)/dx = \ln(1 + x^6) \times 2x = 2x \ln(1 + x^6)$.

For $x \geq 0$: $2x \geq 0$, and $\ln(1 + x^6) > 0$ for $x > 0 \Rightarrow f'(x) > 0$ for $x > 0$.
Hence, f(x) is strictly increasing on (0, ∞) and non-decreasing on [0, ∞).

Answer: $f'(x) = 2x \ln(1 + x^6)$, and f(x) is increasing for $x > 0$.

Problem 2

Evaluate the integral: $I = \int_1^\infty (x + 2)/(x^3\sqrt{(x^2 + 1)})\, dx$.

Solution:

Split the integrand:
$I = \int_1^\infty 1/(x^2\sqrt{(x^2 + 1)})\, dx + \int_1^\infty 2/(x^3\sqrt{(x^2 + 1)})\, dx = I_1 + I_2$.

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$ and $\sqrt{(x^2 + 1)} = \sec\theta$.
Then limits: $x = 1 \Rightarrow \theta = \pi/4$; $x \to \infty \Rightarrow \theta \to \pi/2$.

For I_1:
$I_1 = \int (\sec\theta / \tan^2\theta)\, d\theta = \int \cos\theta / \sin^2\theta\, d\theta$.
Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta \Rightarrow I_1 = -1/\sin\theta = -\csc\theta$.

$I_1 = [-\csc\theta]_{(\pi/4)}^{(\pi/2)} = \sqrt{2} - 1.$

For I_2:
$I_2 = 2\int (\sec\theta / \tan^3\theta)\, d\theta = 2\int \cos^2\theta / \sin^3\theta\, d\theta = 2\int (\csc^3\theta - \csc\theta)\, d\theta.$
Now, $\int \csc^3\theta\, d\theta = -\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\ln|\tan(\theta/2)|.$
$\int \csc\theta\, d\theta = \ln|\tan(\theta/2)|.$

So $I_2 = [-\csc\theta\cot\theta - \ln(\tan(\theta/2))]_{(\pi/4)}^{(\pi/2)} = \sqrt{2} + \ln(\sqrt{2} - 1).$
Using $\ln(\sqrt{2} - 1) = -\ln(\sqrt{2} + 1)$, we get $I_2 = \sqrt{2} - \ln(\sqrt{2} + 1).$

Hence, $I = I_1 + I_2 = (\sqrt{2} - 1) + (\sqrt{2} - \ln(\sqrt{2} + 1)) = 2\sqrt{2} - 1 - \ln(\sqrt{2} + 1).$

Answer: $I = 2\sqrt{2} - 1 - \ln(\sqrt{2} + 1).$

Problem 3

Given $e^{(xy)} + x^2y^2 + z^2 = 4$, where $z = z(x, y)$ and $z(1, 1) = 1$, find z_{xy}.

Solution:

Let $F(x, y, z) = e^{(xy)} + x^2y^2 + z^2 - 4 = 0.$
Differentiate partially w.r.t x:
$\partial F/\partial x + \partial F/\partial z * z_x = 0 \Rightarrow y\, e^{(xy)} + 2xy^2 + 2z\, z_x = 0 \Rightarrow z_x = -[y\, e^{(xy)} + 2xy^2]/(2z).$

At (1, 1, 1): $z_x = -[e + 2]/2.$
Similarly, differentiating w.r.t y:
$x\, e^{(xy)} + 2x^2y + 2z\, z_y = 0 \Rightarrow z_y = -[x\, e^{(xy)} + 2x^2y]/(2z).$
At (1, 1, 1): $z_y = -[e + 2]/2.$

Differentiate $(y\, e^{(xy)} + 2xy^2 + 2z\, z_x = 0)$ w.r.t y:
$e^{(xy)} + xy\, e^{(xy)} + 4xy + 2(z_y\, z_x + z\, z_{xy}) = 0.$
$\Rightarrow z_{xy} = -[e^{(xy)} + xy\, e^{(xy)} + 4xy + 2z_y\, z_x]/(2z).$

At (1, 1, 1): Substitute z_x = z_y = -[e + 2]/2.
Numerator = e + e + 4 + $(e + 2)^2/2 = (e^2/2) + 4e + 6$.
Hence, $z_{xy} = -[(e^2/2) + 4e + 6]/2 = -(e^2 + 8e + 12)/4$.

Answer: $z_{xy}(1, 1) = -(e^2 + 8e + 12)/4$.